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DECOMPOSITION OF THE MIXED-MODE *J*-INTEGRAL—REVISITED

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Abstract—In this paper the correct decomposition of three-dimensional stress and displacement fields for decoupling the mixed mode *J*-integral into mode I, II and III is presented. It is believed that this has not been achieved before as certain components for the correct derivation have been missing. It is shown that by using the correct decomposition method, a different area integral for the mode II and III *J*-integrals is obtained. Several test examples are presented to demonstrate the accuracy of the method. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Since the introduction of the *J*-integral as a fracture mechanics parameter by Cherepanov (1967) and Rice (1968), many numerical solutions have been developed. The application of the finite element method (FEM) and the boundary element method (BEM) to the evaluation of the *J*-integral is well established for two-dimensional problems [see Aliabadi and Rooke (1991)]. For three-dimensional problems the *J*-integral has been directly applied to the finite element method by various workers [see Murakami and Sato (1983); Dodds and Read (1990)], however, the evaluation of surface integrals is cumbersome in FE analyses. This led to the modification of the *J*-integral to a domain integral by Nikishkov and Atluri (1987a), in which the *J*-integral is multiplied by a simple function called an "S" function [see de Lorenzi (1982)]. The method is known as equivalent domain integral (EDI) and is computationally appealing as the domain integral is accurately and easily obtained in FE analysis. EDI has been applied to mixed mode problems by Nikishkov and Atluri (1987b) and Shivakumar and Raju (1990) using the decomposition method [see Ishikawa *et al.* (1979)]. In the decomposition method the mode I, II and III *J*-integrals are directly obtained from the mode I, II and III stresses and displacements.

The application of the *J*-integral method to BEM was recently presented by Aliabadi (1990) for two-dimensional problems and by Rigby and Aliabadi (1993) for three-dimensional problems. Later, Huber *et al.* (1993) presented the formulation for problems involving mode I and III. The BEM is ideally suited to the evaluation of the *J*-integral since the required stresses, strain and derivatives of strain are accurately obtained at internal points in the body given the surface displacements and tractions. These internal point solutions utilize boundary integral equations and, hence, no discretization of the domain is required. The *J*-integral is then calculated by integrating stress, strain and derivative of strain products found from the internal points along a contour in a plane perpendicular to the crack front and also over the area enclosed by the contour. Hence, the *J*-integral is accurately calculated without altering the surface mesh.

To obtain the mode I, II, III *J*-integrals the stress and strain products and stress and derivative of strain products are combined from points symmetric to the crack plane. These integrands are then integrated over the symmetric contour and area enclosed by that

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contour to yield two parts of the J-integral: one comprising of symmetric elastic fields (J^S) and the other comprising of anti-symmetric elastic fields (J^{AS}). The integral J^S is equal to the mode I J-integral, whereas the integral J^{AS} contains both the mode II, III J-integrals. The decomposition method further decouples the mode II, III stresses, strains and derivatives of strain. The stress intensity factors are then obtained directly from the mode I, II, III J-integrals.

In this paper, the proper derivation of the decomposition method for mixed mode J-integral in the three-dimensional boundary element method is presented. It is believed that this has not been obtained before, as some of the components for a correct derivation have been missing. The first of these is the proper decomposition of stress and strain into their mode I, II and III constituents. The equation used in previous papers [see e.g. Nikishkov and Atluri (1987b); Shivakumar and Raju (1990); Rigby and Aliabadi (1993); Huber *et al.* (1993)] is shown to be incorrect. Another component is a different area integral for the mode II and III J-integrals to that quoted in Rigby and Aliabadi (1993) and Huber *et al.* (1993). These two components are significant for the mode II and III J-integrals derived by the decomposition method.

2. THE J-INTEGRAL

Here the energy momentum tensor of Eshelby (1970) is obtained for elastic or nonlinear elastic materials, and from this the three-dimensional J-integral is derived. The strain energy density W in linear elastostatics is defined as:

$$W = W(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} \quad i,j = 1,2,3 \tag{1}$$

where σ_{ij} is the stress tensor and ε_{ij} is the strain tensor with components

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{2}$$

Here $u_{i,j}$ denotes the derivatives of displacements u_i with respect to crack coordinates x_j shown in Fig. 1.

Differentiating the strain energy density W with respect to x_k gives

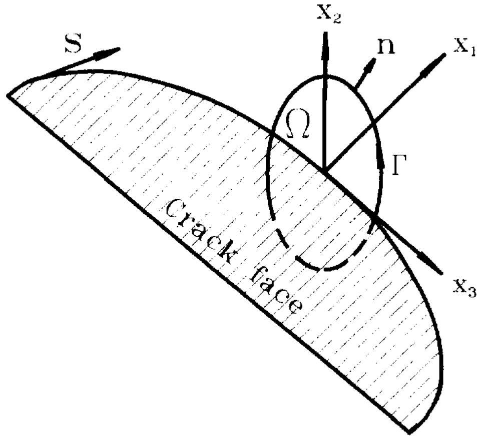

Fig. 1. Crack coordinate system and J contour.

$$\frac{\partial W}{\partial x_k} = \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_k}$$

$$= \frac{1}{2}\sigma_{ij}\left(\frac{\partial u_{i,j}}{\partial x_k} + \frac{\partial u_{j,i}}{\partial x_k}\right) \quad i,j,k = 1,2,3 \tag{3}$$

using eqn (2). Consider

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) = \frac{\partial \sigma_{ij}}{\partial x_j}\frac{\partial u_i}{\partial x_k} + \sigma_{ij}\frac{\partial u_{i,j}}{\partial x_k}. \tag{4}$$

From equilibrium $\partial\sigma_{ij}/\partial x_j = 0$; so substituting eqn (4) into eqn (3) yields

$$\frac{\partial W}{\partial x_k} = \frac{1}{2}\left[\frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u_j}{\partial x_k}\right)\right]$$

$$= \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) \tag{5}$$

as $\sigma_{ij} = \sigma_{ji}$ by equilibrium of moments. Equation (5) can be rewritten in a more compact form as

$$\frac{\partial P_{kj}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(W\delta_{kj} - \sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) = 0 \tag{6}$$

where δ_{kj} is the Kronecker delta function and $[W\delta_{kj} - \sigma_{ij}(\partial u_i/\partial x_k)]$ is Eshelby's momentum tensor and is denoted by P_{kj} [see Amestoy *et al.* (1981)]. All parameters in this tensor are in terms of the crack coordinate system.

Consider a cross-section of a crack shown in Fig. 2. Integrating $P_{kj,j}$ over any area Ω in the plane $x_3 = 0$, whilst excluding the crack singularity, gives

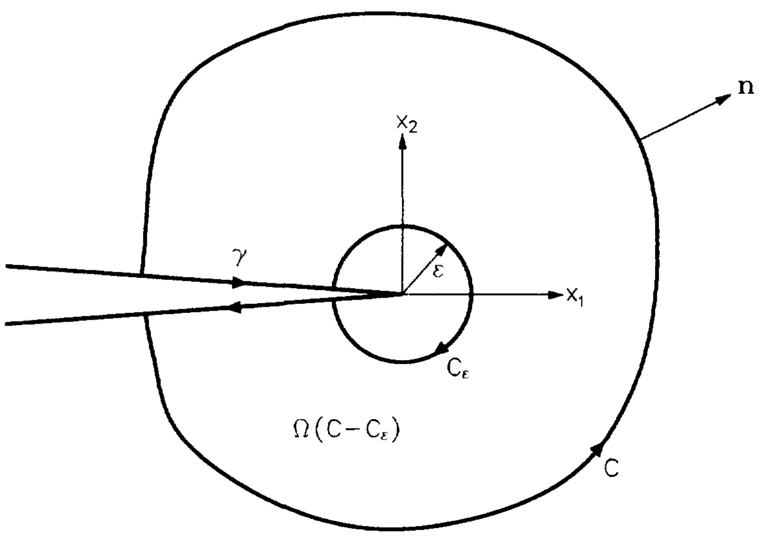

Fig. 2. Contour perpendicular to crack front at s, in the plane $x_3 = 0$.

$$\int_{\Omega(C-C\varepsilon)} \frac{\partial}{\partial x_j}\left(W\delta_{kj} - \sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) \mathrm{d}\Omega = 0 \tag{7}$$

where $\Omega(C-C\varepsilon) = \Omega(C) - \Omega(C\varepsilon)$ denotes the area delimited by the contours C, C_ε and crack surface γ. According to Green's theorem

$$\int_\Gamma T\,\mathrm{d}x_1 + Q\,\mathrm{d}x_2 = \int_\Omega \left(\frac{\partial Q}{\partial x_1} - \frac{\partial T}{\partial x_2}\right)\mathrm{d}\Omega. \tag{8}$$

Since $\mathrm{d}x_1 = -n_2\,\mathrm{d}\Gamma$ and $\mathrm{d}x_2 = n_1\,\mathrm{d}\Gamma$ (where $\mathbf{n}$ is the normal to the contour Γ) one obtains from eqn (7)

$$\int_\Gamma \left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C-C\varepsilon)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega = 0 \tag{9}$$

where Γ is the contour around area $\Omega(C-C\varepsilon)$ and is, therefore, given by

$$\Gamma = C + C_\varepsilon + \gamma. \tag{10}$$

Therefore, eqn (9) can be reordered as

$$\begin{aligned}\int_{C+\gamma} \left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega \\ = -\int_{C_\varepsilon} \left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C\varepsilon)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega.\end{aligned} \tag{11}$$

Consider C_ε to be a circular contour of radius ε. As $\varepsilon \to 0$ then it is generally held that the area term on the right-hand side will tend to zero [see Dodds and Read (1990)]. From eqn (11) the J-integral $J_k(s)$ is defined to be

$$\begin{aligned}J_k(s) &= \int_{\Gamma_\varepsilon} \left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma \\ &= \int_{C+\gamma} \left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega\end{aligned} \tag{12}$$

where Γ_ε is identical to contour C_ε except that it proceeds in an anticlockwise direction. Here $J_k(s)$ is dependent on the position of the crack front s and is defined on the plane $x_3 = 0$. Taking $k = 1$ for a traction free crack, then the contour integrand over crack faces γ is zero as $t_i = \sigma_{ij}n_j = 0$ and $n_1 = 0$. Therefore, one obtains from eqn (12)

$$\begin{aligned}J_1(s) &= \int_{\Gamma_\varepsilon} \left(Wn_1 - \sigma_{ij}\frac{\partial u_i}{\partial x_1}n_j\right)\mathrm{d}\Gamma \\ &= \int_C \left(Wn_1 - \sigma_{ij}\frac{\partial u_i}{\partial x_1}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_1}\right)\mathrm{d}\Omega.\end{aligned} \tag{13}$$

Consider contour Γ_ε held constant. Then the right-hand side of eqn (13) is constant for any contour C, i.e. the right-hand side is path-area independent. The path-area independency of $J_1(s)$ is limited to small regions around position s on the crack front. If parts of the path are distant from the crack front, then $J_1(s)$ is influenced by the singular fields of the points neighbouring s on the crack front. The J-integral is path-area independent in a global sense, i.e. the total strength of the singularity of the whole crack front is independent of the surface enclosing it [see Shivakumar and Raju (1990)].

2.1. *Mixed mode* J*-integral*

For each of the three modes of fracture there is a corresponding J-integral, J^{I}, J^{II} and J^{III} and these are related to J_k as follows [see Cherepanov (1979)]:

$$J_1 = J^{\mathrm{I}} + J^{\mathrm{II}} + J^{\mathrm{III}} \tag{14}$$

$$J_2 = -2\sqrt{J^{\mathrm{I}} J^{\mathrm{II}}}. \tag{15}$$

The J-integrals J^M for each mode $M = \mathrm{I}$, II and III are defined as:

$$J^M = \oint_{\Gamma_\varepsilon} \left(W^M n_1 - \sigma_{ij}^M \frac{\partial u_i^M}{\partial x_1} n_j \right) \mathrm{d}\Gamma, \quad M = \mathrm{I}, \mathrm{II}, \mathrm{III} \tag{16}$$

where Γ_ε is a contour normal to the plane $x_3 = 0$ of vanishing radius ε which proceeds in the anti-clockwise direction. Here σ_{ij}^M and u_i^M are the mode M stresses and displacements, respectively (see Section 2.3) and $W^M = \int_0^{\varepsilon} \sigma_{ij}^M \, \mathrm{d}(\varepsilon_{ij}^M)$. The path-area independence of J^M will be discussed in Section 2.3 and eqn (14) is proven in that section.

One can also define the G^{III} integral [see Nikishkov and Atluri (1987a)]

$$G^{\mathrm{III}} = \oint_{\Gamma_\varepsilon} \left(W_3 n_1 - \sigma_{3j} \frac{\partial u_3}{\partial x_1} n_j \right) \mathrm{d}\Gamma \tag{17}$$

where Γ_ε is a contour of vanishing radius ε in the $x_3 = 0$ plane, perpendicular to the crack front and

$$W_3 = \int_0^{\varepsilon_{3j}} \sigma_{3j} \, \mathrm{d}\varepsilon_{3j}. \tag{18}$$

For linear elasticity, the relationship between J_1 and the mode I, II and III stress intensity factors can be obtained by substituting the three dimensional stress fields in the Appendix into eqn (13). This yields:

$$\begin{aligned} J_1 &= J^{\mathrm{I}} + J^{\mathrm{II}} + J^{\mathrm{III}} \\ &= \frac{1}{E^*}(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2) + \frac{1}{2\mu} K_{\mathrm{III}}^2 \end{aligned} \tag{19}$$

where E^* equals Young's modulus E for plane stress, $E^* = E/(1-\nu^2)$ for plane strain and μ is the shear modulus. Note that a plane stress or plane strain assumption is required to obtain stress intensity factors from the J-integral.

Equations (14)–(19) can be used to obtain the mode I, II and III stress intensity factors from J_1, J_2 and G^{III} as follows:

$$K_{\mathrm{I}} = \tfrac{1}{2}\sqrt{E^*}\left(\sqrt{J_1 - J_2 - G^{\mathrm{III}}} + \sqrt{J_1 + J_2 - G^{\mathrm{III}}}\right) \tag{20}$$

$$K_{\mathrm{II}} = \tfrac{1}{2}\sqrt{E^*}\left(\sqrt{J_1 - J_2 - G^{\mathrm{III}}} - \sqrt{J_1 + J_2 - G^{\mathrm{III}}}\right) \tag{21}$$

$$K_{\mathrm{III}} = \sqrt{2\mu G^{\mathrm{III}}}. \tag{22}$$

However, the use of J_2 leads to numerical difficulties as it will involve integration of singular elastic fields over the crack surface. Also Herrmann and Herrmann (1981) found that J_2 is only path independent, in the same way as J_1, if the integral of Wn_2 over the crack faces vanishes (e.g. if the field stresses σ_{11} and σ_{22} are equal).

There is another approach which avoids the use of J_2. The first step is to split the J_1 integral into two parts:

$$J_1 = J^{S} + J^{AS} \tag{23}$$

where J^S is found from the symmetric elastic fields about the crack plane whereas J^{AS} utilizes the anti-symmetric elastic fields. This is demonstrated in the next section. The integral J^S is equal to J^{I} as the mode I elastic fields are symmetric about the crack plane. This leaves $J^{AS} = J^{II} + J^{III}$, from which J^{II} and J^{III} are decoupled by the decomposition method (Section 2.3). Equation (19) is then used to calculate the mode I, II and III stress intensity factors.

2.2. *Derivation of symmetric and anti-symmetric components*

In this section the symmetric and anti-symmetric components of the J-integral in eqn (23) are derived. These components are obtained from the symmetric and anti-symmetric stresses and strains. Hence, these fields are presented first.

Consider points $P(a, b, c)$ and $P'(a, -b, c)$ which are symmetric about the crack plane $x_2 = 0$ (see Fig. 3). For any arbitrary deformation, the stresses at points $P(a, b, c)$ and $P'(a, -b, c)$ can be expressed in terms of symmetric and anti-symmetric components as follows:

$$\begin{Bmatrix} \sigma_{11P} \\ \sigma_{12P} \\ \sigma_{13P} \\ \sigma_{22P} \\ \sigma_{23P} \\ \sigma_{33P} \end{Bmatrix} = \begin{Bmatrix} \sigma^{S}_{11P} \\ \sigma^{S}_{12P} \\ \sigma^{S}_{13P} \\ \sigma^{S}_{22P} \\ \sigma^{S}_{23P} \\ \sigma^{S}_{33P} \end{Bmatrix} + \begin{Bmatrix} \sigma^{AS}_{11P} \\ \sigma^{AS}_{12P} \\ \sigma^{AS}_{13P} \\ \sigma^{AS}_{22P} \\ \sigma^{AS}_{23P} \\ \sigma^{AS}_{33P} \end{Bmatrix} \tag{24}$$

and

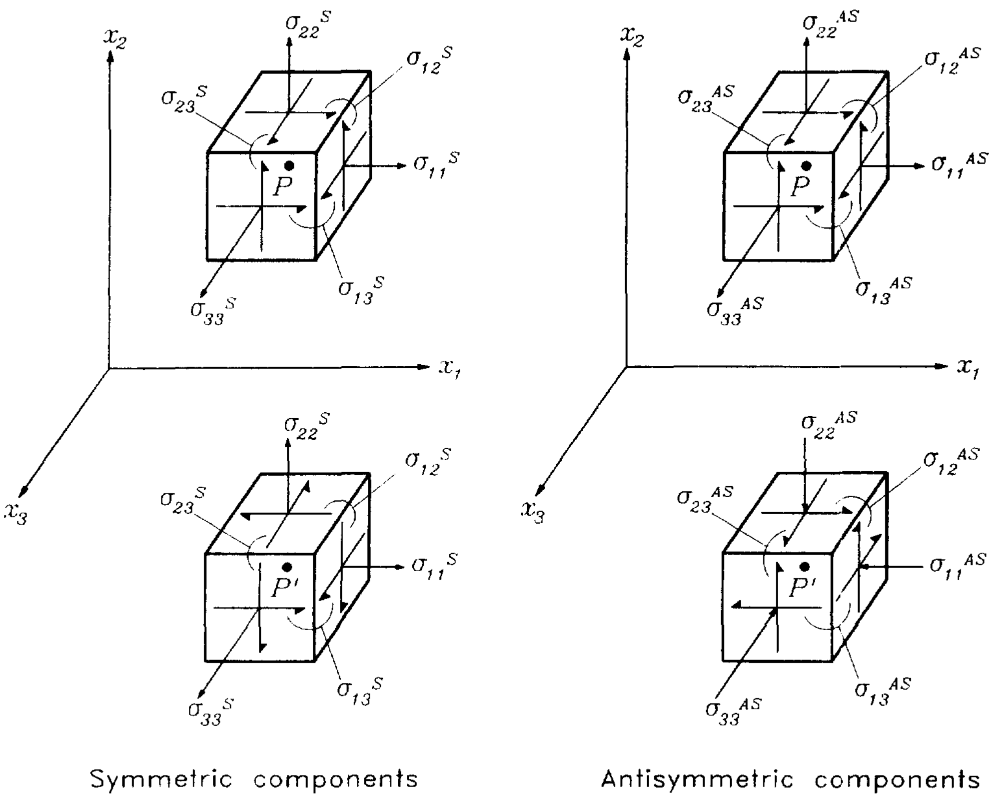

Fig. 3. Symmetric and anti-symmetric components of stress at points P and P'.

$$\begin{Bmatrix} \sigma_{11P'} \\ \sigma_{12P'} \\ \sigma_{13P'} \\ \sigma_{22P'} \\ \sigma_{23P'} \\ \sigma_{33P'} \end{Bmatrix} = \begin{Bmatrix} \sigma^{S}_{11P'} \\ -\sigma^{S}_{12P'} \\ \sigma^{S}_{13P'} \\ \sigma^{S}_{22P'} \\ -\sigma^{S}_{23P'} \\ \sigma^{S}_{33P'} \end{Bmatrix} + \begin{Bmatrix} -\sigma^{AS}_{11P'} \\ \sigma^{AS}_{12P'} \\ -\sigma^{AS}_{13P'} \\ -\sigma^{AS}_{22P'} \\ \sigma^{AS}_{23P'} \\ -\sigma^{AS}_{33P'} \end{Bmatrix} \tag{25}$$

where S and AS denote symmetric and anti-symmetric components. Therefore, the symmetric components of stress are given as:

$$\begin{Bmatrix} \sigma^{S}_{11} \\ \sigma^{S}_{12} \\ \sigma^{S}_{13} \\ \sigma^{S}_{22} \\ \sigma^{S}_{23} \\ \sigma^{S}_{33} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \sigma_{11P}+\sigma_{11P'} \\ \sigma_{12P}-\sigma_{12P'} \\ \sigma_{13P}+\sigma_{13P'} \\ \sigma_{22P}+\sigma_{22P'} \\ \sigma_{23P}-\sigma_{23P'} \\ \sigma_{33P}+\sigma_{33P'} \end{Bmatrix} \tag{26}$$

and similarly

$$\begin{Bmatrix} \sigma^{AS}_{11} \\ \sigma^{AS}_{12} \\ \sigma^{AS}_{13} \\ \sigma^{AS}_{22} \\ \sigma^{AS}_{23} \\ \sigma^{AS}_{33} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \sigma_{11P}-\sigma_{11P'} \\ \sigma_{12P}+\sigma_{12P'} \\ \sigma_{13P}-\sigma_{13P'} \\ \sigma_{22P}-\sigma_{22P'} \\ \sigma_{23P}+\sigma_{23P'} \\ \sigma_{33P}-\sigma_{33P'} \end{Bmatrix} \tag{27}$$

for anti-symmetric components. The strains are related to stresses by Hookes' law:

$$\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \delta_{ij}\left(\frac{v}{1+v}\right)\sigma_{kk}\right]$$

where δ_{ij} is the Kronecker delta and μ the shear modulus. Thus the strains can be written as

$$\varepsilon_{ij} = \varepsilon^{S}_{ij} + \varepsilon^{AS}_{ij}$$

$$= \frac{1}{2} \begin{Bmatrix} \varepsilon_{11P}+\varepsilon_{11P'} \\ \varepsilon_{12P}-\varepsilon_{12P'} \\ \varepsilon_{13P}+\varepsilon_{13P'} \\ \varepsilon_{22P}+\varepsilon_{22P'} \\ \varepsilon_{23P}-\varepsilon_{23P'} \\ \varepsilon_{33P}+\varepsilon_{33P'} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} \varepsilon_{11P}-\varepsilon_{11P'} \\ \varepsilon_{12P}+\varepsilon_{12P'} \\ \varepsilon_{13P}-\varepsilon_{13P'} \\ \varepsilon_{22P}-\varepsilon_{22P'} \\ \varepsilon_{23P}+\varepsilon_{23P'} \\ \varepsilon_{33P}-\varepsilon_{33P'} \end{Bmatrix}. \tag{28}$$

The stresses and strains at point P' are now examined. Consider a function

$$\sigma'_{ij}(x_1, x_2, x_3) = \sigma_{ij}(x_1, -x_2, x_3).$$

Then differentiating with respect to x_2 at P', one obtains

$$\left.\frac{\partial \sigma_{ij}(x_1, x_2, x_3)}{\partial x_2}\right|_{(a,-b,c)} = \left.\frac{\partial \sigma_{ij}(x_1, -x_2, x_3)}{\partial(-x_2)}\right|_{(a,b,c)} = -\left.\frac{\partial \sigma'_{ij}(x_1, x_2, x_3)}{\partial x_2}\right|_{(a,b,c)}$$

and so for equilibrium of forces at P'

$$\left.\frac{\partial \sigma_{ij}}{\partial x_j}\right|_{(a,-b,c)} = \left[\frac{\partial \sigma'_{ij}}{\partial x_1} - \frac{\partial \sigma'_{ij}}{\partial x_2} + \frac{\partial \sigma'_{ij}}{\partial x_3}\right]_{(a,b,c)} = 0.$$

Thus from eqns (26) and (27)

$$\frac{\partial \sigma^{S}_{ij}}{\partial x_j} = \frac{\partial \sigma^{AS}_{ij}}{\partial x_j} = 0 \tag{29}$$

i.e. the symmetric and anti-symmetric stress fields exhibit equilibrium of forces.

The displacement derivatives $\partial u_i^{S}/\partial x_j$ and $\partial u_i^{AS}/\partial x_j$ are now obtained from the strains ε_{ij}. An example of this procedure is presented for $\partial u_3^{AS}/\partial x_2$. Starting with

$$\begin{aligned}\varepsilon^{AS}_{23} &= \tfrac{1}{2}(\varepsilon_{23P} + \varepsilon_{23P'}) = \tfrac{1}{2}[\varepsilon_{23}(x_1, x_2, x_3) + \varepsilon_{23}(x_1, -x_2, x_3)]_{(a,b,c)} \\ &= \tfrac{1}{2}(\varepsilon_{23} + \varepsilon'_{23})\end{aligned}$$

then

$$\begin{aligned}\varepsilon'_{23}(x_1, x_2, x_3) &= \varepsilon_{23}(x_1, x_2, x_3)|_{(a,-b,c)} \\ &= \frac{1}{2}\left[\frac{\partial u_2(x_1, x_2, x_3)}{\partial x_3} + \frac{\partial u_3(x_1, x_2, x_3)}{\partial x_2}\right]_{(a,-b,c)} \\ &= \frac{1}{2}\left[\frac{\partial u_2(x_1, -x_2, x_3)}{\partial x_3} + \frac{\partial u_3(x_1, -x_2, x_3)}{\partial(-x_2)}\right]_{(a,b,c)} \\ &= \frac{1}{2}\left[\frac{\partial u'_2(x_1, x_2, x_3)}{\partial x_3} - \frac{\partial u'_3(x_1, x_2, x_3)}{\partial x_2}\right]_{(a,b,c)}\end{aligned}$$

and so

$$\varepsilon^{AS}_{23} = \frac{1}{4}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u'_2}{\partial x_3} - \frac{\partial u'_3}{\partial x_2}\right]_{(a,b,c)}.$$

The symmetric and anti-symmetric strains are related to the displacement derivatives by

$$\varepsilon^{N}_{ij} = \frac{1}{2}\left(\frac{\partial u_i^{N}}{\partial x_j} + \frac{\partial u_j^{N}}{\partial x_i}\right) \tag{30}$$

for small deformation where $N = \mathrm{S}$ or AS. Therefore

$$\varepsilon^{AS}_{23} = \frac{1}{2}\left(\frac{\partial u_2^{AS}}{\partial x_3} + \frac{\partial u_3^{AS}}{\partial x_2}\right)$$

and so

$$\frac{\partial u_3^{AS}}{\partial x_2} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_3'}{\partial x_2}\right).$$

In a similar fashion, all the symmetric and anti-symmetric displacement derivatives can be obtained from eqn (30) and are:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i^S}{\partial x_j} + \frac{\partial u_i^{AS}}{\partial x_j}$$
$$= \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_j} + \dfrac{\partial u_1'}{\partial x_j} \\ \dfrac{\partial u_2}{\partial x_j} - \dfrac{\partial u_2'}{\partial x_j} \\ \dfrac{\partial u_3}{\partial x_j} + \dfrac{\partial u_3'}{\partial x_j} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_j} - \dfrac{\partial u_1'}{\partial x_j} \\ \dfrac{\partial u_2}{\partial x_j} + \dfrac{\partial u_2'}{\partial x_j} \\ \dfrac{\partial u_3}{\partial x_j} - \dfrac{\partial u_3'}{\partial x_j} \end{Bmatrix}. \tag{31}$$

Returning to the J-integral, eqn (13) can be written using the symmetric and anti-symmetric components as

$$J_1 = \int_C \left[\left(\int_0^{\varepsilon_{ij}} (\sigma_{ij}^S + \sigma_{ij}^{AS})\,\mathrm{d}(\varepsilon_{ij}^S + \varepsilon_{ij}^{AS})\right) n_1 - (\sigma_{ij}^S + \sigma_{ij}^{AS}) n_j \frac{\partial}{\partial x_1}(u_i^S + u_i^{AS})\right] \mathrm{d}\Gamma$$
$$- \int_{\Omega(C - C_\varepsilon)} \frac{\partial}{\partial x_3}\left[(\sigma_{i3}^S + \sigma_{i3}^{AS}) \frac{\partial}{\partial x_1}(u_i^S + u_i^{AS})\right] \mathrm{d}\Omega \tag{32}$$

using the definition of W in eqn (1) and including the (zero) area integral for $\Omega(C_\varepsilon)$. By inspecting Fig. 3, the symmetric and anti-symmetric stresses σ_{ij}' at P' are related to the stresses σ_{ij} at P as follows:

$$\begin{Bmatrix} \sigma_{11}'^S \\ \sigma_{12}'^S \\ \sigma_{13}'^S \\ \sigma_{22}'^S \\ \sigma_{23}'^S \\ \sigma_{33}'^S \end{Bmatrix} = \begin{Bmatrix} \sigma_{11}^S \\ -\sigma_{12}^S \\ \sigma_{13}^S \\ \sigma_{22}^S \\ -\sigma_{23}^S \\ \sigma_{33}^S \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{11}'^{AS} \\ \sigma_{12}'^{AS} \\ \sigma_{13}'^{AS} \\ \sigma_{22}'^{AS} \\ \sigma_{23}'^{AS} \\ \sigma_{33}'^{AS} \end{Bmatrix} = \begin{Bmatrix} -\sigma_{11}^{AS} \\ \sigma_{12}^{AS} \\ -\sigma_{13}^{AS} \\ -\sigma_{22}^{AS} \\ \sigma_{23}^{AS} \\ -\sigma_{33}^{AS} \end{Bmatrix}. \tag{33}$$

Similarly for strains:

$$\begin{Bmatrix} \varepsilon_{11}'^S \\ \varepsilon_{12}'^S \\ \varepsilon_{13}'^S \\ \varepsilon_{22}'^S \\ \varepsilon_{23}'^S \\ \varepsilon_{33}'^S \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11}^S \\ -\varepsilon_{12}^S \\ \varepsilon_{13}^S \\ \varepsilon_{22}^S \\ -\varepsilon_{23}^S \\ \varepsilon_{33}^S \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{11}'^{AS} \\ \varepsilon_{12}'^{AS} \\ \varepsilon_{13}'^{AS} \\ \varepsilon_{22}'^{AS} \\ \varepsilon_{23}'^{AS} \\ \varepsilon_{33}'^{AS} \end{Bmatrix} = \begin{Bmatrix} -\varepsilon_{11}^{AS} \\ \varepsilon_{12}^{AS} \\ -\varepsilon_{13}^{AS} \\ -\varepsilon_{22}^{AS} \\ \varepsilon_{23}^{AS} \\ -\varepsilon_{33}^{AS} \end{Bmatrix} \tag{34}$$

and displacement derivatives with respect to x_1:

$$\begin{Bmatrix} u'^{S}_{1,1} \\ u'^{S}_{2,1} \\ u'^{S}_{3,1} \end{Bmatrix} = \begin{Bmatrix} u^{S}_{1,1} \\ -u^{S}_{2,1} \\ u^{S}_{3,1} \end{Bmatrix}, \quad \begin{Bmatrix} u'^{AS}_{1,1} \\ u'^{AS}_{2,1} \\ u'^{AS}_{3,1} \end{Bmatrix} = \begin{Bmatrix} -u^{AS}_{1,1} \\ u^{AS}_{2,1} \\ -u^{AS}_{3,1} \end{Bmatrix}. \tag{35}$$

Consider contour C is symmetric about the crack plane $x_2 = 0$. Then the normal $\mathbf{n}'$ at P' is related to the normal $\mathbf{n}$ at P as follows:

$$(n'_1, n'_2) = (n_1, -n_2).$$

Therefore, from eqns (33)–(35) the products in eqn (32) behave in the following manner:

$$\sigma'^{M}_{ij}\,\mathrm{d}(\varepsilon'^{N}_{ij}) = \pm\sigma^{M}_{ij}\,\mathrm{d}(\varepsilon^{N}_{ij}) \tag{36}$$

$$\sigma'^{M}_{ij} n'_j \frac{\partial u'^{N}_i}{\partial x_1} = \pm\sigma^{M}_{ij} n_j \frac{\partial u^{N}_i}{\partial x_1} \tag{37}$$

$$\sigma'^{M}_{i3} \frac{\partial u'^{N}_i}{\partial x_1} = \pm\sigma^{M}_{i3} \frac{\partial u^{N}_i}{\partial x_1} \tag{38}$$

where M, $N = \mathrm{S}$ or AS and σ_{ij}, ε_{ij}, $\partial u_i/\partial x_1$ are values at point P and σ'_{ij}, ε'_{ij}, $\partial u'_i/\partial x_1$ are values at point P'. In eqns (36)–(38), the positive sign denotes the case of $M = N$ while the negative sign denotes the case $M \neq N$. Thus in the case of $M \neq N$, the integrands in eqn (32) cancel each other at symmetric points about the crack plane. As the contour C is symmetric about the crack plane, then eqn (32) reduces to

$$\begin{aligned} J_1 &= \sum_{N=1}^{2} \int_C \left(W^N n_1 - \sigma^N_{ij} \frac{\partial u^N_i}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C-C_\varepsilon)} \frac{\partial}{\partial x_3} \left(\sigma^N_{i3} \frac{\partial u^N_i}{\partial x_1} \right) \mathrm{d}\Omega \\ &= J^{S} + J^{AS} \end{aligned} \tag{39}$$

where $N = 1$ for symmetric components (S) and $N = 2$ for anti-symmetric components (AS), and $W^N = \int_0^{\varepsilon} \sigma^N_{ij}\,\mathrm{d}(\varepsilon^N_{ij})$.

The path-area independence of J^{S} and J^{AS} is now established. From eqns (13) and (36)–(38)

$$\begin{aligned} J_1 &= \int_{\Gamma_\varepsilon} \left(W n_1 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j \right) \mathrm{d}\Gamma \\ &= \sum_{N=1}^{2} \int_{\Gamma_\varepsilon} \left(W^N n_1 - \sigma^N_{ij} \frac{\partial u^N_i}{\partial x_1} n_j \right) \mathrm{d}\Gamma. \end{aligned} \tag{40}$$

Thus, eqn (39) can be reordered to give

$$\sum_{N=1}^{2} \int_{\Gamma} \left(W^N n_1 - \sigma^N_{ij} \frac{\partial u^N_i}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C-C_\varepsilon)} \frac{\partial}{\partial x_3} \left(\sigma^N_{i3} \frac{\partial u^N_i}{\partial x_1} \right) \mathrm{d}\Omega = 0 \tag{41}$$

where $\Gamma = C + C_\varepsilon + \gamma$. It is now shown that this holds for each N. Applying Green's theorem [eqn (8)] to eqn (41) one obtains, for $N =$ S or AS,

$$\int_{\Gamma}\left(W^N n_1-\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}n_j\right)\mathrm{d}\Gamma-\int_{\Omega(C-C_\varepsilon)}\frac{\partial}{\partial x_3}\left(\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}\right)\mathrm{d}\Omega$$
$$=\int_{\Omega(C-C_\varepsilon)}\left[\frac{\partial W^N}{\partial x_1}-\frac{\partial}{\partial x_j}\left(\sigma_{i3}^N\frac{\partial u_i^N}{\partial x_1}\right)\right]\mathrm{d}\Omega \quad (42)$$

where area $\Omega(C-C_\varepsilon)$ contains no singularities and noting $n_3=0$. From eqn (3)

$$\frac{\partial W^N}{\partial x_1}=\sigma_{ij}^N\frac{\partial \varepsilon_{ij}^N}{\partial x_1}=\sigma_{ij}^N\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial u_i^N}{\partial x_j}+\frac{\partial u_j^N}{\partial x_i}\right)$$
$$=\sigma_{ij}^N\frac{\partial^2 u_i^N}{\partial x_1\,\partial x_j} \quad (43)$$

as $\sigma_{ij}^N=\sigma_{ji}^N$. Also

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}\right)=\sigma_{ij}^N\frac{\partial^2 u_i^N}{\partial x_j\,\partial x_1} \quad (44)$$

as $\sigma_{ij,j}^N=0$. It can be easily shown from eqn (31) that

$$\frac{\partial^2 u_i^N}{\partial x_j\,\partial x_1}=\frac{\partial^2 u_i^N}{\partial x_1\,\partial x_j} \quad (45)$$

and so from eqns (43) and (44)

$$\int_{\Omega(C-C_\varepsilon)}\left[\frac{\partial W^N}{\partial x_1}-\frac{\partial}{\partial x_j}\left(\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}\right)\right]\mathrm{d}\Omega=0. \quad (46)$$

Therefore, by reordering eqn (42)

$$\int_{C}\left(W^N n_1-\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}n_j\right)\mathrm{d}\Gamma-\int_{\Omega(C)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}^N\frac{\partial u_i^N}{\partial x_1}\right)\mathrm{d}\Omega$$
$$=-\int_{C_\varepsilon}\left(W^N n_1-\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}n_j\right)\mathrm{d}\Gamma-\int_{\Omega(C_\varepsilon)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}^N\frac{\partial u_i^N}{\partial x_1}\right)\mathrm{d}\Omega \quad (47)$$

as the contour integrand is zero along γ as tractions $t_i^N=\sigma_{ij}^N n_j=0$ and $n_1=0$. If the contour C_ε is kept constant, then the left-hand side of eqn (47) is constant for any symmetric contour C. It is, therefore, path-area independent. It has already been noted that the integral over area $\Omega(C_\varepsilon)$ tends to zero as $\varepsilon\to 0$. Therefore, one can obtain from eqn (47) the path-area independent integral:

$$J^N:=\int_{\Gamma_\varepsilon}\left(W^N n_1-\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}n_j\right)\mathrm{d}\Gamma$$
$$=\int_{C}\left(W^N n_1-\sigma_{ij}^N\frac{\partial u_i^N}{\partial x_1}n_j\right)\mathrm{d}\Gamma-\int_{\Omega(C)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}^N\frac{\partial u_i^N}{\partial x_1}\right)\mathrm{d}\Omega \quad (48)$$

for $N=$ S or AS. Here the contour Γ_ε is identical to the contour C_ε except that it proceeds in an anti-clockwise direction. The mode I J-integral is given by J^{S}, whereas J^{AS} is related to both the mode II and III J-integrals:

$$J^{AS} = J^{II} + J^{III}.$$

The decomposition method for decoupling the mode II and III terms from the J^{AS} integral is now presented.

2.3. *Decomposition method*

In this method the stresses, strains and derivatives of strain are decoupled into their mode I, II and III components. This enables the mode I, II and III J-integrals to be obtained. The mode I stresses are the same as the symmetric stresses [eqn (26)], whereas the anti-symmetric stresses in eqn (27) are split into mode II and III.

2.3.1. *Re-evaluation of stress decomposition.* The decomposition of stress has been given by various authors [e.g. Nikishkov and Atluri (1987a); Shivakumar and Raju (1990); Rigby and Aliabadi (1993); Huber *et al.* (1993)] as:

$$\sigma_{ij} = \sigma_{ij}^{I} + \sigma_{ij}^{II} + \sigma_{ij}^{III}$$

where the right-hand side has been quoted as being equal to

$$\frac{1}{2}\begin{Bmatrix} \sigma_{11}+\sigma'_{11} \\ \sigma_{12}-\sigma'_{12} \\ \sigma_{13}+\sigma'_{13} \\ \sigma_{22}+\sigma'_{22} \\ \sigma_{23}-\sigma'_{23} \\ \sigma_{33}+\sigma'_{33} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \sigma_{11}-\sigma'_{11} \\ \sigma_{12}+\sigma'_{12} \\ 0 \\ \sigma_{22}-\sigma'_{22} \\ 0 \\ 0 \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \sigma_{13}-\sigma'_{13} \\ 0 \\ \sigma_{23}+\sigma'_{23} \\ \sigma_{33}-\sigma'_{33} \end{Bmatrix}.$$

In this section this equation is re-evaluated in the light of the three-dimensional stress equations near the crack front given in the Appendix. It is found that the expressions quoted for σ_{ij}^{II} and σ_{ij}^{III} are incorrect.

In the Appendix the singular stresses and the constant stresses in the vicinity of the crack front are listed. It can be seen that the σ_{33} stress is zero for mode III in the vicinity of the crack front and that the singular stresses are functions of $\sin(a\theta)$ and $\cos(b\theta)$ where a and b are constants. Therefore, because of the properties of $\sin(a\theta)$ and $\cos(b\theta)$, one can relate the stresses σ'_{ij} at P' to the stresses σ_{ij} at P for each mode I, II or III. Thus, Table 1 is obtained from the Appendix.

Referring to Fig. 3, it can be seen how the stresses in Table 1 relate to the symmetric and anti-symmetric stresses. For example, from Table 1 the mode I σ'_{12} at P' is equal to minus the σ_{12} at P, i.e. mode I $\sigma'_{12} = -\sigma_{12}$. From Fig. 3 the symmetric σ_{12}^{S} behaves in the same fashion, i.e. $\sigma_{12P'}^{S} = -\sigma_{12P}^{S}$. Therefore, $\sigma_{12}^{I} = \sigma_{12}^{S}$ and similarly

Table 1. Relationship between stresses σ_{ij} at P and σ'_{ij} at P' for mode I, II and III

Mode I	Mode II	Mode III	Non singular
$\sigma'_{11} = \sigma_{11}$	$\sigma'_{11} = -\sigma_{11}$		
$\sigma'_{12} = -\sigma_{12}$	$\sigma'_{12} = \sigma_{12}$		
		$\sigma'_{13} = -\sigma_{13}$	$\sigma'_{13} = \sigma_{13}$
$\sigma'_{22} = \sigma_{22}$	$\sigma'_{22} = -\sigma_{22}$		
		$\sigma'_{23} = \sigma_{23}$	$\sigma'_{23} = -\sigma_{23} = 0$
$\sigma'_{33} = \sigma_{33}$	$\sigma'_{33} = -\sigma_{33}$		

$$\sigma_{ij} = \sigma^{\mathrm{I}}_{ij}+\sigma^{\mathrm{II}}_{ij}+\sigma^{\mathrm{III}}_{ij}$$

$$= \begin{Bmatrix} \sigma^{\mathrm{S}}_{11} \\ \sigma^{\mathrm{S}}_{12} \\ \sigma^{\mathrm{S}}_{13} \\ \sigma^{\mathrm{S}}_{22} \\ \sigma^{\mathrm{S}}_{23} \\ \sigma^{\mathrm{S}}_{33} \end{Bmatrix} + \begin{Bmatrix} \sigma^{\mathrm{AS}}_{11} \\ \sigma^{\mathrm{AS}}_{12} \\ 0 \\ \sigma^{\mathrm{AS}}_{22} \\ 0 \\ \sigma^{\mathrm{AS}}_{33} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \sigma^{\mathrm{AS}}_{13} \\ 0 \\ \sigma^{\mathrm{AS}}_{23} \\ 0 \end{Bmatrix}. \tag{49}$$

The non-singular stress σ_{13} behaves in the same way as σ^{S}_{13}, i.e. non-singular $\sigma'_{13} = \sigma_{13}$ and $\sigma^{\mathrm{S}}_{13P'} = \sigma^{\mathrm{S}}_{13P}$. The singular stress $\sigma^{\mathrm{III}}_{13}$, $\sigma^{\mathrm{III}}_{23}$ are both anti-symmetric and the non-singular stresses σ_{13}, σ_{23} are both symmetric. As the mode I stresses are considered to be symmetric, then the non-singular σ_{13}, σ_{23} are included in mode I.

Therefore, from eqns (26), (27) and (49), the resulting stress decomposition is:

$$\sigma_{ij} = \sigma^{\mathrm{I}}_{ij}+\sigma^{\mathrm{II}}_{ij}+\sigma^{\mathrm{III}}_{ij}$$

$$= \frac{1}{2}\begin{Bmatrix} \sigma_{11}+\sigma'_{11} \\ \sigma_{12}-\sigma'_{12} \\ \sigma_{13}+\sigma'_{13} \\ \sigma_{22}+\sigma'_{22} \\ \sigma_{23}-\sigma'_{23} \\ \sigma_{33}+\sigma'_{33} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \sigma_{11}-\sigma'_{11} \\ \sigma_{12}+\sigma'_{12} \\ 0 \\ \sigma_{22}-\sigma'_{22} \\ 0 \\ \sigma_{33}-\sigma'_{33} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \sigma_{13}-\sigma'_{13} \\ 0 \\ \sigma_{23}+\sigma'_{23} \\ 0 \end{Bmatrix}. \tag{50}$$

It can be seen that the term $\sigma_{33}-\sigma'_{33}$ has swapped from the $\sigma^{\mathrm{III}}_{ij}$ expression to the $\sigma^{\mathrm{II}}_{ij}$ expression compared with the decomposition equation used in Nikishkov and Atluri (1987b). An example of the effect of the incorrect decomposition equation can be seen in e.g. Rigby and Aliabadi (1993), where the mode II and in particular mode III results for a penny-shaped crack have been adversely affected (see e.g. decomposition results in Fig. 7 of that reference). Equation (50) enables the proper decomposition of the J^{AS} integral into its mode II and III components, and this reflected in the accurate penny-shaped crack results presented in Section 4.

2.3.2. *Strain and displacement components.* The strains are related to stresses by Hookes' Law:

$$\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij}-\delta_{ij}\left(\frac{v}{1-v}\right)\sigma_{kk}\right] \tag{51}$$

where δ_{ij} is the Kronecker delta and μ the shear modulus. The decomposition of the strains can be achieved by simply inserting eqn (50) into eqn (51), for example

$$\begin{aligned}\varepsilon^{\mathrm{II}}_{11} &= \frac{1}{4\mu}\left[\sigma_{11}-\sigma'_{11}-\left(\frac{v}{1+v}\right)(\sigma_{kk}-\sigma'_{kk})\right] \\ &= \tfrac{1}{2}(\varepsilon_{11}-\varepsilon'_{11}).\end{aligned} \tag{52}$$

Therefore, the strains can be written as

$$\varepsilon_{ij} = \varepsilon_{ij}^{\mathrm{I}} + \varepsilon_{ij}^{\mathrm{II}} + \varepsilon_{ij}^{\mathrm{III}}$$
$$= \begin{Bmatrix} \varepsilon_{11}+\varepsilon'_{11} \\ \varepsilon_{12}-\varepsilon'_{12} \\ \varepsilon_{13}+\varepsilon'_{13} \\ \varepsilon_{22}+\varepsilon'_{22} \\ \varepsilon_{23}-\varepsilon'_{23} \\ \varepsilon_{33}+\varepsilon'_{33} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \varepsilon_{11}-\varepsilon'_{11} \\ \varepsilon_{12}+\varepsilon'_{12} \\ 0 \\ \varepsilon_{22}-\varepsilon'_{22} \\ 0 \\ \varepsilon_{33}-\varepsilon'_{33} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \varepsilon_{13}-\varepsilon'_{13} \\ 0 \\ \varepsilon_{23}+\varepsilon'_{23} \\ 0 \end{Bmatrix}. \tag{53}$$

As strains are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

then from eqn (53)

$$\varepsilon_{ij} = \varepsilon_{ij}^{\mathrm{I}} + \varepsilon_{ij}^{\mathrm{II}} + \varepsilon_{ij}^{\mathrm{III}}$$
$$= \frac{1}{4}\begin{Bmatrix} 2\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u'_1}{\partial x_1}\right) \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u'_1}{\partial x_2} - \dfrac{\partial u'_2}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u'_1}{\partial x_3} + \dfrac{\partial u'_3}{\partial x_1} \\ 2\left(\dfrac{\partial u_2}{\partial x_2} - \dfrac{\partial u'_2}{\partial x_2}\right) \\ \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u'_2}{\partial x_3} + \dfrac{\partial u'_3}{\partial x_2} \\ 2\left(\dfrac{\partial u_3}{\partial x_3} + \dfrac{\partial u'_3}{\partial x_3}\right) \end{Bmatrix} + \frac{1}{4}\begin{Bmatrix} 2\left(\dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u'_1}{\partial x_1}\right) \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u'_1}{\partial x_2} + \dfrac{\partial u'_2}{\partial x_1} \\ 0 \\ 2\left(\dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u'_2}{\partial x_2}\right) \\ 0 \\ 2\left(\dfrac{\partial u_3}{\partial x_3} - \dfrac{\partial u'_3}{\partial x_3}\right) \end{Bmatrix}$$
$$+ \frac{1}{4}\begin{Bmatrix} 0 \\ 0 \\ \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u'_1}{\partial x_3} - \dfrac{\partial u'_3}{\partial x_1} \\ 0 \\ \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u'_2}{\partial x_3} - \dfrac{\partial u'_3}{\partial x_2} \\ 0 \end{Bmatrix}. \tag{54}$$

From eqn (54) the mode I, II, III derivatives of displacement are given as

$$\frac{\partial u_i}{\partial x_1} = \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} + \frac{\partial u_i^{\mathrm{II}}}{\partial x_1} + \frac{\partial u_i^{\mathrm{III}}}{\partial x_1}$$
$$= \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u'_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u'_2}{\partial x_1} \\ \dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u'_3}{\partial x_1} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u'_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u'_2}{\partial x_1} \\ 0 \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u'_3}{\partial x_1} \end{Bmatrix} \tag{55}$$

$$\frac{\partial u_i}{\partial x_2} = \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_1'}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_2}-\dfrac{\partial u_2'}{\partial x_2} \\ \dfrac{\partial u_3}{\partial x_2}+\dfrac{\partial u_3'}{\partial x_2} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_2}-\dfrac{\partial u_1'}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_2}+\dfrac{\partial u_2'}{\partial x_2} \\ 0 \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \dfrac{\partial u_3}{\partial x_2}-\dfrac{\partial u_3'}{\partial x_2} \end{Bmatrix} \tag{56}$$

$$\frac{\partial u_i}{\partial x_3} = \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_3}+\dfrac{\partial u_1'}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_3}-\dfrac{\partial u_2'}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_3}+\dfrac{\partial u_3'}{\partial x_3} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \dfrac{\partial u_3}{\partial x_3}-\dfrac{\partial u_3'}{\partial x_3} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_3}-\dfrac{\partial u_1'}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_3}+\dfrac{\partial u_2'}{\partial x_3} \\ 0 \end{Bmatrix}. \tag{57}$$

It is now shown that $J_1 = J^{\mathrm{I}}+J^{\mathrm{II}}+J^{\mathrm{III}}$. It has already been demonstrated in Section 2.2, eqn (40) that

$$\begin{aligned} J_1 &= J^{\mathrm{S}}+J^{\mathrm{AS}} \\ &= \int_{\Gamma_\varepsilon}\left[\left(\int_0^{\varepsilon_{ij}}\sigma_{ij}^{\mathrm{S}}\,\mathrm{d}(\varepsilon_{ij}^{\mathrm{S}})\right)n_1-\sigma_{ij}^{\mathrm{S}}\frac{\partial u_i^{\mathrm{S}}}{\partial x_1}n_j\right]\mathrm{d}\Gamma \\ &\quad+\int_{\Gamma_\varepsilon}\left[\left(\int_0^{\varepsilon_{ij}}\sigma_{ij}^{\mathrm{AS}}\,\mathrm{d}(\varepsilon_{ij}^{\mathrm{AS}})\right)n_1-\sigma_{ij}^{\mathrm{AS}}\frac{\partial u_i^{\mathrm{AS}}}{\partial x_1}n_j\right]\mathrm{d}\Gamma. \end{aligned} \tag{58}$$

The symmetric fields are mode I, whereas the anti-symmetric fields are mode II plus mode III. Therefore,

$$J^{\mathrm{AS}} = \int_{\Gamma_\varepsilon}\left[\int_0^{\varepsilon_{ij}}(\sigma_{ij}^{\mathrm{II}}+\sigma_{ij}^{\mathrm{III}})\,\mathrm{d}(\varepsilon_{ij}^{\mathrm{II}}+\varepsilon_{ij}^{\mathrm{III}})\right]n_1\,\mathrm{d}\Gamma-\int_{\Gamma_\varepsilon}(\sigma_{ij}^{\mathrm{II}}+\sigma_{ij}^{\mathrm{III}})\frac{\partial(u_i^{\mathrm{II}}+u_i^{\mathrm{III}})}{\partial x_1}n_j\,\mathrm{d}\Gamma. \tag{59}$$

However, there are no cross-products between stress and strain for mode II and mode III from eqns (50), (53) and (55), i.e.

$$\begin{aligned} \sigma_{ij}^{M}\varepsilon_{ij}^{N} &= 0, \quad M \neq N \\ \sigma_{ij}^{M}\frac{\partial u_i^{N}}{\partial x_1}n_j &= 0, \quad M \neq N \end{aligned} \tag{60}$$

where M, $N = \mathrm{II}$, III and $n_3 = 0$. Thus

$$\begin{aligned} J^{\mathrm{AS}} &= J^{\mathrm{II}}+J^{\mathrm{III}} \\ &= \sum_{M=\mathrm{II}}^{\mathrm{III}}\int_{\Gamma_\varepsilon}\left[\left(\int_0^{\varepsilon_{ij}}\sigma_{ij}^{M}\,\mathrm{d}(\varepsilon_{ij}^{M})\right)n_1-\sigma_{ij}^{M}\frac{\partial u_i^{M}}{\partial x_1}n_j\right]\mathrm{d}\Gamma \end{aligned} \tag{61}$$

and $J_1 = J^{\mathrm{I}}+J^{\mathrm{II}}+J^{\mathrm{III}}$ where J^M is defined by eqn (16).

The mode I, II and III J-integrals must be expressed in terms of the contour C to be calculated by the boundary element method. To obtain the J^M integrals from the mode I, II, III stresses, strains and derivatives of strain, it is necessary to start from eqn (3) with k set to 1 :

$$\frac{\partial W}{\partial x_1}-\sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}=0. \tag{62}$$

Consider a cross-section of a crack shown in Fig. 2 where the contour Γ is defined as $\Gamma = C+C_\varepsilon+\gamma$. Integrating eqn (62) over the area $\Omega(C-C_\varepsilon)$ which is bounded by Γ and free from singularities:

$$\int_{\Omega(C-C\varepsilon)}\left(\frac{\partial W}{\partial x_1}-\sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}\right)\mathrm{d}\Omega=0. \tag{63}$$

Using Green's theorem [eqn (8)]

$$\int_\Gamma W n_1\,\mathrm{d}\Gamma-\int_{\Omega(C-C_\varepsilon)}\sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}\mathrm{d}\Omega=0. \tag{64}$$

This can be written in terms of symmetric (i.e. mode I) and anti-symmetric (i.e. mode II + III) components as

$$\int_{\Gamma_\varepsilon}\left[\int_0^{\varepsilon_{ij}}(\sigma_{ij}^{\mathrm{S}}+\sigma_{ij}^{\mathrm{AS}})\,\mathrm{d}(\varepsilon_{ij}^{\mathrm{S}}+\varepsilon_{ij}^{\mathrm{AS}})\right]n_1\,\mathrm{d}\Gamma-\int_{\Omega(C-C_\varepsilon)}(\sigma_{ij}^{\mathrm{S}}+\sigma_{ij}^{\mathrm{AS}})\frac{\partial(\varepsilon_{ij}^{\mathrm{S}}+\varepsilon_{ij}^{\mathrm{AS}})}{\partial x_1}\mathrm{d}\Omega=0. \tag{65}$$

However, it has already been shown that products $\sigma_{ij}^M\,\mathrm{d}(\varepsilon_{ij}^N)$ cancel out for $M\neq N$ and a symmetric contour Γ [see eqn (36)]. Thus, eqn (65) reduces to

$$\sum_{N=1}^{2}\int_\Gamma W^N n_1\,\mathrm{d}\Gamma-\int_{\Omega(C-C_\varepsilon)}\sigma_{ij}^N\frac{\partial \varepsilon_{ij}^N}{\partial x_1}\mathrm{d}\Omega=0 \tag{66}$$

where $N = 1$ for symmetric components and $N = 2$ for anti-symmetric components, and $W^N=\int_0^{\varepsilon_{ij}}\sigma_{ij}^N\,\mathrm{d}(\varepsilon_{ij}^N)$. It is easily shown that this equation holds for each N. Applying Green's theorem to eqn (66) one obtains, for N = S or AS,

$$\begin{aligned}\int_\Gamma W^N n_1\,\mathrm{d}\Gamma-\int_{\Omega(C-C_\varepsilon)}\sigma_{ij}^N\frac{\partial \varepsilon_{ij}^N}{\partial x_1}\mathrm{d}\Omega&=\int_{\Omega(C-C_\varepsilon)}\left(\frac{\partial W^N}{\partial x_1}-\sigma_{ij}^N\frac{\partial \varepsilon_{ij}^N}{\partial x_1}\right)\mathrm{d}\Omega\\&=0\end{aligned} \tag{67}$$

as

$$\frac{\partial W^N}{\partial x_1}=\sigma_{ij}^N\frac{\partial \varepsilon_{ij}^N}{\partial x_1}.$$

For N = AS, eqn (67) can be further decomposed into mode II and III components as there are no cross-products between stress and strain for mode II and III. Therefore,

$$\int_\Gamma W^M n_1\,\mathrm{d}\Gamma-\int_{\Omega(C-C_\varepsilon)}\sigma_{ij}^M\frac{\partial \varepsilon_{ij}^M}{\partial x_1}\mathrm{d}\Omega=0\quad M=\mathrm{I,II,III} \tag{68}$$

holds for each mode M. Here $\Gamma = C+C_\varepsilon+\gamma$ (see Fig. 2). Concentrating on the area integral in eqn (68):

$$\int_{\Omega(C-C_\varepsilon)} \sigma_{ij}^M \frac{\partial \varepsilon_{ij}^M}{\partial x_1} \mathrm{d}\Omega = \int_{\Omega(C-C_\varepsilon)} \sigma_{ij}^M \frac{\partial^2 u_i^M}{\partial x_1 \partial x_j} \mathrm{d}\Omega$$

$$= \int_{\Omega(C-C_\varepsilon)} \left(\sigma_{i1}^M \frac{\partial^2 u_i^M}{\partial x_1 \partial x_1} + \sigma_{i2}^M \frac{\partial^2 u_i^M}{\partial x_1 \partial x_2} + \sigma_{i3}^M \frac{\partial^2 u_i^M}{\partial x_1 \partial x_3} \right) \mathrm{d}\Omega \tag{69}$$

as $\sigma_{ij}^M = \sigma_{ji}^M$ and using eqn (2). However

$$\frac{\partial}{\partial x_1}\left(\sigma_{i1}^M \frac{\partial u_i^M}{\partial x_1}\right) = \frac{\partial \sigma_{i1}^M}{\partial x_1}\frac{\partial u_i^M}{\partial x_1} + \sigma_{i1}^M \frac{\partial^2 u_i^M}{\partial x_1 \partial x_1} \tag{70}$$

$$\frac{\partial}{\partial x_2}\left(\sigma_{i2}^M \frac{\partial u_i^M}{\partial x_1}\right) = \frac{\partial \sigma_{i2}^M}{\partial x_2}\frac{\partial u_i^M}{\partial x_1} + \sigma_{i2}^M \frac{\partial^2 u_i^M}{\partial x_2 \partial x_1}. \tag{71}$$

Now from eqns (55) and (56) the following relationship is obtained:

$$\frac{\partial}{\partial x_2}\left(\frac{\partial u_i^M}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial u_i^M}{\partial x_2}\right) \tag{72}$$

whereas it is noted from eqns (55) and (57) that

$$\frac{\partial}{\partial x_3}\left(\frac{\partial u_i^M}{\partial x_1}\right) \neq \frac{\partial}{\partial x_1}\left(\frac{\partial u_i^M}{\partial x_3}\right) \quad M = \mathrm{II}, \mathrm{III}. \tag{73}$$

Substituting eqns (70) and (71) into eqn (69) and using relationship (72), gives:

$$\int_{\Omega(C-C_\varepsilon)} \sigma_{ij}^M \frac{\partial \varepsilon_{ij}^M}{\partial x_1} \mathrm{d}\Omega = \int_{\Omega(C-C_\varepsilon)} \left[\frac{\partial}{\partial x_1}\left(\sigma_{i1}^M \frac{\partial u_i^M}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\sigma_{i2}^M \frac{\partial u_i^M}{\partial x_1}\right)\right] \mathrm{d}\Omega$$

$$+ \int_{\Omega(C-C_\varepsilon)} \left[\sigma_{i3}^M \frac{\partial^2 u_i^M}{\partial x_1 \partial x_3} - \left(\frac{\partial \sigma_{i1}^M}{\partial x_1} + \frac{\partial \sigma_{i2}^M}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}\right] \mathrm{d}\Omega. \tag{74}$$

If $M = \mathrm{I}$ then $\partial \sigma_{ij}^{\mathrm{I}}/\partial x_j = \partial \sigma_{ij}^{\mathrm{S}}/\partial x_j = 0$ and so $-(\partial \sigma_{i1}^{\mathrm{I}}/\partial x_1 + \partial \sigma_{i2}^{\mathrm{I}}/\partial x_2)$ can be replaced by $\partial \sigma_{i3}^{\mathrm{I}}/\partial x_3$. For mode II or III then comparing like terms in eqns (50) and (27) results in

$$\frac{\partial \sigma_{ij}^{\mathrm{AS}}}{\partial x_j} = \frac{\partial \sigma_{i1}^{\mathrm{AS}}}{\partial x_1} + \frac{\partial \sigma_{i2}^{\mathrm{AS}}}{\partial x_2} + \frac{\partial \sigma_{i3}^{\mathrm{AS}}}{\partial x_3} = 0$$

$$= \frac{\partial}{\partial x_1}\begin{Bmatrix} \sigma_{11}^{\mathrm{II}} \\ \sigma_{21}^{\mathrm{II}} \\ \sigma_{31}^{\mathrm{III}} \end{Bmatrix} + \frac{\partial}{\partial x_2}\begin{Bmatrix} \sigma_{12}^{\mathrm{II}} \\ \sigma_{22}^{\mathrm{II}} \\ \sigma_{32}^{\mathrm{III}} \end{Bmatrix} + \frac{\partial}{\partial x_3}\begin{Bmatrix} \sigma_{13}^{\mathrm{III}} \\ \sigma_{23}^{\mathrm{III}} \\ \sigma_{33}^{\mathrm{II}} \end{Bmatrix}. \tag{75}$$

Reordering eqn (75) yields

$$\frac{\partial \sigma_{11}^{\mathrm{II}}}{\partial x_1} + \frac{\partial \sigma_{12}^{\mathrm{II}}}{\partial x_2} = -\frac{\partial \sigma_{13}^{\mathrm{III}}}{\partial x_3} \tag{76}$$

$$\frac{\partial \sigma_{21}^{\mathrm{II}}}{\partial x_1} + \frac{\partial \sigma_{22}^{\mathrm{II}}}{\partial x_2} = -\frac{\partial \sigma_{23}^{\mathrm{III}}}{\partial x_3} \tag{77}$$

$$\frac{\partial \sigma_{31}^{\mathrm{III}}}{\partial x_1}+\frac{\partial \sigma_{32}^{\mathrm{III}}}{\partial x_2}=-\frac{\partial \sigma_{33}^{\mathrm{II}}}{\partial x_3}. \tag{78}$$

It is also noted from eqn (50)

$$\frac{\partial \sigma_{11}^{\mathrm{III}}}{\partial x_1}+\frac{\partial \sigma_{12}^{\mathrm{III}}}{\partial x_2}=0 \tag{79}$$

$$\frac{\partial \sigma_{21}^{\mathrm{III}}}{\partial x_1}+\frac{\partial \sigma_{22}^{\mathrm{III}}}{\partial x_2}=0 \tag{80}$$

$$\frac{\partial \sigma_{31}^{\mathrm{II}}}{\partial x_1}+\frac{\partial \sigma_{32}^{\mathrm{II}}}{\partial x_2}=0. \tag{81}$$

Hence the components $(\partial \sigma_{i1}^{M}/\partial x_1+\partial \sigma_{i2}^{M}/\partial x_2)$ can always be replaced by $\partial \sigma_{13}^{\mathrm{III}}/\partial x_3$, $\partial \sigma_{23}^{\mathrm{III}}/\partial x_3$, $\partial \sigma_{33}^{\mathrm{II}}/\partial x_3$ or zero for mode II or III.

Applying Green's theorem to eqn (74) yields

$$\int_{\Omega(C-C_\varepsilon)} \sigma_{ij}^{M}\frac{\partial \varepsilon_{ij}^{M}}{\partial x_1}\mathrm{d}\Omega=\int_{\Gamma}\sigma_{ij}^{M}\frac{\partial u_i^M}{\partial x_1}n_j\,\mathrm{d}\Gamma+\int_{\Omega(C-C_\varepsilon)}\sigma_{i3}^{M}\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\mathrm{d}\Omega$$
$$-\int_{\Omega(C-C_\varepsilon)}\left(\frac{\partial \sigma_{i1}^{M}}{\partial x_1}+\frac{\partial \sigma_{i2}^{M}}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}\mathrm{d}\Omega \tag{82}$$

as $n_3=0$. Combining eqn (82) with eqn (68) yields

$$\int_{\Gamma}\left(W^M n_1-\sigma_{ij}^{M}\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma+\int_{\Omega(C-C_\varepsilon)}\left[\left(\frac{\partial \sigma_{i1}^{M}}{\partial x_1}+\frac{\partial \sigma_{i2}^{M}}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}-\sigma_{i3}^{M}\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\right]\mathrm{d}\Omega=0. \tag{83}$$

The contour Γ can be separated into contours C, C_ε and γ. The contour integral is zero on γ as $n_1=0$ and tractions $t_i^M=\sigma_{ij}^M n_j$ are assumed to be zero. The area integrand is of the type

$$I_{\mathrm{A}}=\frac{\partial \sigma_{i3}}{\partial x_3}\frac{\partial u_i}{\partial x_1}+\sigma_{i3}\frac{\partial^2 u_i}{\partial x_1\,\partial x_3}$$

noting eqns (76)–(81). The area integral over Ω_ε will tend to zero as $\varepsilon\to 0$ as I_{A} is similar to $\partial/\partial x_3[\sigma_{i3}(\partial u_i/\partial x_1)]$. Replacing C_ε by its anti-clockwise equivalent Γ_ε and utilizing the definition of J^M in eqn (16) yields

$$J^M=\int_{\Gamma_\varepsilon}\left(W^M n_1-\sigma_{ij}^{M}\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma$$
$$=\int_{C}\left(W^M n_1-\sigma_{ij}^{M}\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma+\int_{\Omega(C)}\left(\frac{\partial \sigma_{i1}^{M}}{\partial x_1}+\frac{\partial \sigma_{i2}^{M}}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}\mathrm{d}\Omega$$
$$-\int_{\Omega(C)}\sigma_{i3}^{M}\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\mathrm{d}\Omega\quad M=\mathrm{I},\mathrm{II},\mathrm{III}. \tag{84}$$

The area integral in eqn (84) is different from that quoted in previous papers [see e.g. Rigby and Aliabadi (1993); Huber *et al.* (1993)]. The area integral has been quoted as

$$\int_{\Omega(C)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}^M \frac{\partial u_i^M}{\partial x_1}\right) \mathrm{d}\Omega.$$

This area integral is erroneous from eqns (50) and (55) as

$$\sigma_{i3}\frac{\partial u_i}{\partial x_1} \neq \sum_{M=1}^{3} \sigma_{i3}^M \frac{\partial u_i^M}{\partial x_1}$$

which would result in $J_1 \neq J^{\mathrm{I}} + J^{\mathrm{II}} + J^{\mathrm{III}}$.

It is noted from eqn (73) that $\partial^2 u_i^M/\partial x_1\,\partial x_3 \neq \partial^2 u_i^M/\partial x_3\,\partial x_1$ for $M = \mathrm{II}$ and III. The term $\partial^2 u_i/\partial x_1\,\partial x_3$ is obtained by differentiating $\partial u_i/\partial x_3$ with respect to x_1. Hence, from eqn (57):

$$\begin{aligned}\frac{\partial}{\partial x_1}\left(\frac{\partial u_i}{\partial x_3}\right) &= \frac{\partial^2 u_i}{\partial x_1\,\partial x_3}\\ &= \frac{1}{2}\left\{\begin{matrix} \dfrac{\partial^2 u_1}{\partial x_1\,\partial x_3} + \dfrac{\partial^2 u_1'}{\partial x_1\,\partial x_3} \\ \dfrac{\partial^2 u_2}{\partial x_1\,\partial x_3} - \dfrac{\partial^2 u_2'}{\partial x_1\,\partial x_3} \\ \dfrac{\partial^2 u_3}{\partial x_1\,\partial x_3} + \dfrac{\partial^2 u_3'}{\partial x_1\,\partial x_3} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} 0 \\ 0 \\ \dfrac{\partial^2 u_3}{\partial x_1\,\partial x_3} - \dfrac{\partial^2 u_3'}{\partial x_1\,\partial x_3} \end{matrix}\right\}\\ &\quad + \frac{1}{2}\left\{\begin{matrix} \dfrac{\partial^2 u_1}{\partial x_1\,\partial x_3} - \dfrac{\partial^2 u_1'}{\partial x_1\,\partial x_3} \\ \dfrac{\partial^2 u_2}{\partial x_1\,\partial x_3} + \dfrac{\partial^2 u_2'}{\partial x_1\,\partial x_3} \\ 0 \end{matrix}\right\}. \end{aligned} \tag{85}$$

The mode I, II and III J-integrals can be found from eqn (84). The stress intensity factors can be related to the J-integral by using eqn (19):

$$\begin{aligned} J &= J^{\mathrm{I}} + J^{\mathrm{II}} + J^{\mathrm{III}} \\ &= \frac{1}{E^*}(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2) + \frac{1}{2\mu} K_{\mathrm{III}}^2 \end{aligned}$$

where E^* equals Young's modulus E for equivalent plane stress and $E^* = E/(1-\nu^2)$ for equivalent plane strain and μ is the shear modulus.

2.3.3. *Path-area independence of* J^{M}. Path-area independence is investigated by integrating eqn (6) (with $k = 1$) over an area containing no singularities for each mode. This integral is equal to zero for symmetric and anti-symmetric stress fields. Using Green's theorem:

$$\begin{aligned}\int_{\Omega(C-C_\varepsilon)} \frac{\partial P_{1j}^M}{\partial x_j} \mathrm{d}\Omega &= \int_{\Omega(C-C_\varepsilon)} \left[\frac{\partial W^M}{\partial x_1} - \frac{\partial}{\partial x_j}\left(\sigma_{ij}^M \frac{\partial u_i^M}{\partial x_1}\right)\right] \mathrm{d}\Omega \\ &= \int_\Gamma \left(W^M n_1 - \sigma_{ij}^M \frac{\partial u_i^M}{\partial x_1} n_j\right) \mathrm{d}\Gamma - \int_{\Omega(C-C_\varepsilon)} \frac{\partial}{\partial x_3}\left(\sigma_{i3}^M \frac{\partial u_i^M}{\partial x_1}\right) \mathrm{d}\Omega \end{aligned} \tag{86}$$

for $M = \mathrm{I, II, III}$ where $\Gamma = C + C_\varepsilon + \gamma$ (see Fig. 2) and $\Omega(C - C_\varepsilon)$ is the area bounded by Γ and free from singularities. The area integral on the left-hand side becomes

$$\int_{\Omega(C-C_\varepsilon)} \left[\sigma_{ij}^M \frac{\partial \varepsilon_{ij}^M}{\partial x_1} - \frac{\partial \sigma_{ij}^M}{\partial x_j} \frac{\partial u_i^M}{\partial x_1} - \sigma_{ij}^M \frac{\partial^2 u_i^M}{\partial x_j \partial x_1} \right] \mathrm{d}\Omega$$

$$= \int_{\Omega(C-C_\varepsilon)} \left[\sigma_{ij}^M \left(\frac{\partial^2 u_i^M}{\partial x_1 \partial x_j} - \frac{\partial^2 u_i^M}{\partial x_j \partial x_1} \right) - \frac{\partial \sigma_{ij}^M}{\partial x_j} \frac{\partial u_i^M}{\partial x_1} \right] \mathrm{d}\Omega \quad (87)$$

using eqn (3) and $\sigma_{ij}^M = \sigma_{ji}^M$. This integral is zero for mode I as $\sigma_{ij,j}^{\mathrm{I}} = 0$ and $\partial^2 u_i^{\mathrm{I}} / \partial x_1 \partial x_j = \partial^2 u_i^{\mathrm{I}} / \partial x_j \partial x_1$ [see eqns (50), (55)–(57)]. Thus, reordering the right-hand side of eqn (86):

$$\int_C \left(W^{\mathrm{I}} n_1 - \sigma_{ij}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C)} \frac{\partial}{\partial x_3} \left(\sigma_{i3}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} \right) \mathrm{d}\Omega$$

$$= -\int_{C_\varepsilon} \left(W^{\mathrm{I}} n_1 - \sigma_{ij}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C_\varepsilon)} \frac{\partial}{\partial x_3} \left(\sigma_{i3}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} \right) \mathrm{d}\Omega \quad (88)$$

as the contour integrand is zero on γ for a traction free crack and $n_1 = 0$. For constant C_ε, the left-hand side is constant for any C. Hence the left-hand side is path-area independent. Also the integrand over area $\Omega(C_\varepsilon)$ will tend to zero as noted before. Thus, one obtains

$$J^{\mathrm{I}} = \int_{\Gamma_\varepsilon} \left(W^{\mathrm{I}} n_1 - \sigma_{ij}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} n_j \right) \mathrm{d}\Gamma$$

$$= \int_C \left(W^{\mathrm{I}} n_1 - \sigma_{ij}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C)} \frac{\partial}{\partial x_3} \left(\sigma_{i3}^{\mathrm{I}} \frac{\partial u_i^{\mathrm{I}}}{\partial x_1} \right) \mathrm{d}\Omega \quad (89)$$

where J^{I} is path-area independent and the contour Γ_ε is identical to contour C_ε except that it proceeds in the anti-clockwise direction.

For mode II and III, the area integral in eqn (87) becomes

$$\int_{\Omega(C-C_\varepsilon)} \left[\sigma_{i3}^M \left(\frac{\partial^2 u_i^M}{\partial x_1 \partial x_3} - \frac{\partial^2 u_i^M}{\partial x_3 \partial x_1} \right) - \frac{\partial \sigma_{ij}^M}{\partial x_j} \frac{\partial u_i^M}{\partial x_1} \right] \mathrm{d}\Omega \neq 0 \quad (90)$$

as $\partial^2 u_i^M / \partial x_1 \partial x_j = \partial^2 u_i^M / \partial x_j \partial x_1$ for $j = 1, 2$, but not for $j = 3$ [see eqns (55)–(57)] and also $\sigma_{ij,j}^M \neq 0$ for $M = \mathrm{II, III}$, as can be seen in eqns (50). This area integral is equal to the left-hand side of eqn (86) and so

$$\int_\Gamma \left(W^M n_1 - \sigma_{ij}^M \frac{\partial u_i^M}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C-C_\varepsilon)} \frac{\partial}{\partial x_3} \left(\sigma_{i3}^M \frac{\partial u_i^M}{\partial x_1} \right) \mathrm{d}\Omega$$

$$= \int_{\Omega(C-C_\varepsilon)} \left[\sigma_{i3}^M \left(\frac{\partial^2 u_i^M}{\partial x_1 \partial x_3} - \frac{\partial^2 u_i^M}{\partial x_3 \partial x_1} \right) - \frac{\partial \sigma_{ij}^M}{\partial x_j} \frac{\partial u_i^M}{\partial x_1} \right] \mathrm{d}\Omega. \quad (91)$$

Therefore

$$\int_{\Gamma}\left(W^M n_1-\sigma_{ij}^M\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma+\int_{\Omega(C-C_\varepsilon)}\left[\left(\frac{\partial\sigma_{i1}^M}{\partial x_1}+\frac{\partial\sigma_{i2}^M}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}-\sigma_{i3}^M\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\right]\mathrm{d}\Omega=0 \quad (92)$$

which is eqn (83). Splitting the contour integral into two parts

$$\int_{C}\left(W^M n_1-\sigma_{ij}^M\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma+\int_{\Omega(C)}\left[\left(\frac{\partial\sigma_{i1}^M}{\partial x_1}+\frac{\partial\sigma_{i2}^M}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}-\sigma_{i3}^M\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\right]\mathrm{d}\Omega$$

$$=-\int_{C_\varepsilon}\left(W^M n_1-\sigma_{ij}^M\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma+\int_{\Omega(C_\varepsilon)}\left[\left(\frac{\partial\sigma_{i1}^M}{\partial x_1}+\frac{\partial\sigma_{i2}^M}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}-\sigma_{i3}^M\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\right]\mathrm{d}\Omega \quad (93)$$

as the contour integrand is zero on γ for a traction free crack and $n_1=0$. Consider C_ε held constant. Then the left-hand side of eqn (93) is path-area independent as it has the same value for any C. Also the area integrand is of the type

$$I_A=\frac{\partial\sigma_{i3}}{\partial x_3}\frac{\partial u_i}{\partial x_1}+\sigma_{i3}\frac{\partial^2 u_i}{\partial x_1\,\partial x_3}$$

noting eqns (76)–(81). The area integral over Ω_ε will tend to zero as $\varepsilon\to 0$ as I_A is similar to $\partial/\partial x_3[\sigma_{i3}(\partial u_i/\partial x_1)]$. Therefore

$$J^M=\int_{\Gamma_\varepsilon}\left(W^M n_1-\sigma_{ij}^M\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma$$

$$=\int_{C}\left(W^M n_1-\sigma_{ij}^M\frac{\partial u_i^M}{\partial x_1}n_j\right)\mathrm{d}\Gamma$$

$$+\int_{\Omega(C)}\left[\left(\frac{\partial\sigma_{i1}^M}{\partial x_1}+\frac{\partial\sigma_{i2}^M}{\partial x_2}\right)\frac{\partial u_i^M}{\partial x_1}-\sigma_{i3}^M\frac{\partial^2 u_i^M}{\partial x_1\,\partial x_3}\right]\mathrm{d}\Omega$$

is path-area independent. Here the contour Γ_ε is identical to C_ε except it proceeds in an anti-clockwise direction. It is noted that this is identical to eqn (84).

For modes II and III, $\int_\Omega P_{kj,j}^M\,\mathrm{d}\Omega\neq 0$ from eqns (86) and (90) where P_{kj} is Eshelby's momentum tensor. Also $\sigma_{ij,j}^M\neq 0$ for $M=\mathrm{II}$, III. Both of these expressions represent equilibrium of forces in the stress field. Thus the mode II or III stress fields are not in equilibrium individually. However, the sum of mode II and III constitutes the anti-symmetric field which is in equilibrium. Thus $\int_\Omega P_{kj,j}^{\mathrm{II+III}}\,\mathrm{d}\Omega=0$ and $\sigma_{ij,j}^{\mathrm{II+III}}=0$.

3. THE BOUNDARY ELEMENT METHOD

The displacement values $u_i(\mathbf{X}')$ at an interior point $\mathbf{X}'$ are given by

$$u_i(\mathbf{X}')=\int_\Gamma U_{ij}(\mathbf{X}',\mathbf{x})t_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x})-\int_\Gamma T_{ij}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \quad (94)$$

The terms $U_{ij}(\mathbf{X}',\mathbf{x})$ and $T_{ij}(\mathbf{X}',\mathbf{x})$ are the fundamental displacement and traction solutions, respectively, for a point force in an infinite domain (i.e. Kelvin's solution), u_j and t_j are the known values of displacement and traction on the boundary Γ. The strain field throughout the body may be obtained by differentiation of the above equation to yield (assuming zero body forces):

$$u_{i,k}(\mathbf{X}') = \int_\Gamma U_{ij,k}(\mathbf{X}',\mathbf{x})t_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma T_{ij,k}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \tag{95}$$

where i, j range from 1 to 3 and $\mathbf{x}$ is a point on the surface Γ. The stresses at interior point $\mathbf{X}'$ are given by

$$\sigma_{ij}(\mathbf{X}') = \int_\Gamma D_{kij}(\mathbf{X}',\mathbf{x})t_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma S_{kij}(\mathbf{X}',\mathbf{x})u_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \tag{96}$$

The area integrals defined in eqns (48) and (84) require two more boundary integral equations to give $\partial^2 u_j/\partial x_1\,\partial x_3$ and $\partial\sigma_{3j}/\partial x_3$. As $\partial^2 u_j/\partial x_1\,\partial x_3 = \partial^2 u_j/\partial x_3\,\partial x_1$ then this kernel is obtained by differentiating eqn (95) (with k set to 1) with respect to x_3 and $\partial\sigma_{3j}/\partial x_3$ is obtained by differentiating eqn (96) (with i set to 3) with respect to x_3. Hence:

$$u_{i,31}(\mathbf{X}') = \int_\Gamma U_{ij,31}(\mathbf{X}',\mathbf{x})t_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma T_{ij,31}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \tag{97}$$

and

$$\sigma_{3j,3}(\mathbf{X}') = \int_\Gamma D_{k3j,3}(\mathbf{X}',\mathbf{x})t_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma S_{k3j,3}(\mathbf{X}',\mathbf{x})u_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \tag{98}$$

4. TEST EXAMPLES

The test examples establish the accuracy and reliability of the proposed J-integral technique for the embedded penny-shaped crack in an infinite domain, for which the exact solution is available. Path-area independence of the J-integral is investigated.

4.1. *Inclined penny-shaped crack*

A crack at the centre of a large cylinder of radius d and height $2h$ (see Fig. 4) was modelled. The stress intensity factors for this configuration should be very close to those of an inclined penny-shaped crack under remote tensile loading in an infinite elastic isotropic body. The crack has a radius a and is inclined at an angle of α to the x_1 axis of the cylinder. The loading is a tensile stress of magnitude σ applied on the ends of the cylinder. The crack length to cylinder radius ratio was chosen as $a/d = 0.1$ and the height to radius ratio as $h/d = 1.5$. The Poisson's ratio was chosen as $\nu = 0.3$.

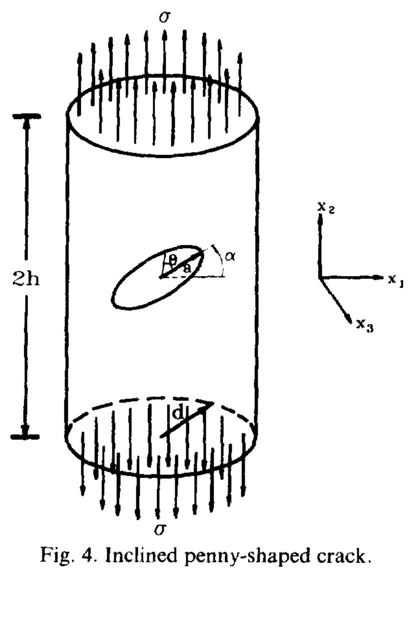

Fig. 4. Inclined penny-shaped crack.

Table 2. Results for penny-shaped crack, $\alpha = 0^\circ$

Mesh density	r/a	K_I^*	K_I^*% error	K_{II}^*	K_{III}^*
Coarse	0.25	1.0126	1.3	7.0×10^{-8}	4.7×10^{-8}
Coarse	0.5	0.9907	−0.9	1.4×10^{-7}	9.3×10^{-8}
Coarse	0.75	0.9812	−1.9	2.1×10^{-7}	1.5×10^{-7}
Fine	0.25	0.9927	−0.7	1.5×10^{-6}	2.7×10^{-7}
Fine	0.5	0.9969	−0.3	1.6×10^{-6}	3.7×10^{-7}
Fine	0.75	0.9922	−0.8	1.6×10^{-6}	5.1×10^{-7}

4.1.1. *Numerical results—non-inclined crack.* For $\alpha = 0^\circ$ both K_{II} and K_{III} are zero and $K_I^* = 1.0$ where $K_I^* = K_I/K_0$ [see Tada *et al.* (1985)]. Here K_I^* has been normalized by the stress intensity factor K_0 for a penny-shaped crack in an infinite body:

$$K_0 = \frac{2}{\pi}\sigma\sqrt{\pi a}.$$

Two boundary element models were set up: the first model has two regions, 120 elements per region and 760 nodes and the second is a finer mesh with two regions, 168 elements per region and 1144 nodes. The J contour to crack radius ratios of $r/a = 0.25$, 0.5 and 0.75 were tried. The minimum element side length on the crack faces was $0.5a$ for the coarse mesh and $0.25a$ for the fine mesh. The results for $\theta = 30^\circ$ are shown in Table 2.

The K_I^* results are very accurate for $r/a = 0.25$ and 0.5, and not much advantage is gained by using a fine mesh rather than a coarse mesh. However, for $r/a = 0.75$ the accuracy of the J-integral results are more sensitive to mesh density. The effect of the coarse mesh on the accuracy of the results (which is most pronounced for $r/a = 0.75$) demonstrates that a certain degree of mesh density is required to reasonably approximate the displacement and traction fields around the crack front.

It can be seen that the path-area independency of the J-integral is maintained at various radii. If the ratio $r/a \leqslant 0.5$ then good accuracy is obtained regardless of fineness of mesh employed. This indicates the robustness of the J-integral method. It is also noted that the ratio of the area part of the J-integral to the contour part was approximately 0.067 for $r/a = 0.25$ and 0.176 for $r/a = 0.5$.

4.1.2. *Numerical results—inclined crack.* Next the crack is inclined at an angle of 30° then 45°. For each configuration two mesh densities were employed of the same density as those used for the non-inclined crack. The J contour to crack radius ratios of $r/a = 0.3$, 0.5 and 0.75 were used for each r/a ratio. The maximum percentage error for the stress intensity factors compared to the theoretical solution from Tada *et al.* (1985) is presented in Table 3 for the 45° configuration. The percentage errors for the 30° configuration are slightly smaller.

Table 3. Percentage errors for penny-shaped crack, $\alpha = 45^\circ$

Mesh density	r/a	Max K_I^* % error	Decomposition % error Max K_{II}^* $\theta = 0^\circ$	Max K_{III}^* $\theta = 90^\circ$
Coarse	0.3	1.2	0.2	2.5
Coarse	0.5	−2.1	−2.2	−1.3
Coarse	0.75	−1.7	−0.5	2.6
Fine	0.3	−1.3	−0.7	−0.2
Fine	0.5	−0.3	−0.2	2.1
Fine	0.75	0.9	0.7	9.5

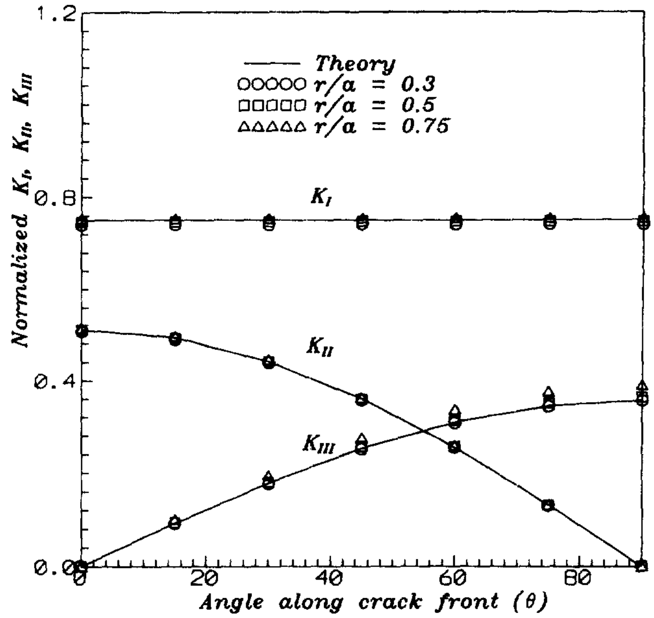

Fig. 5. Penny-shaped crack at 30° to tensile stress, fine mesh.

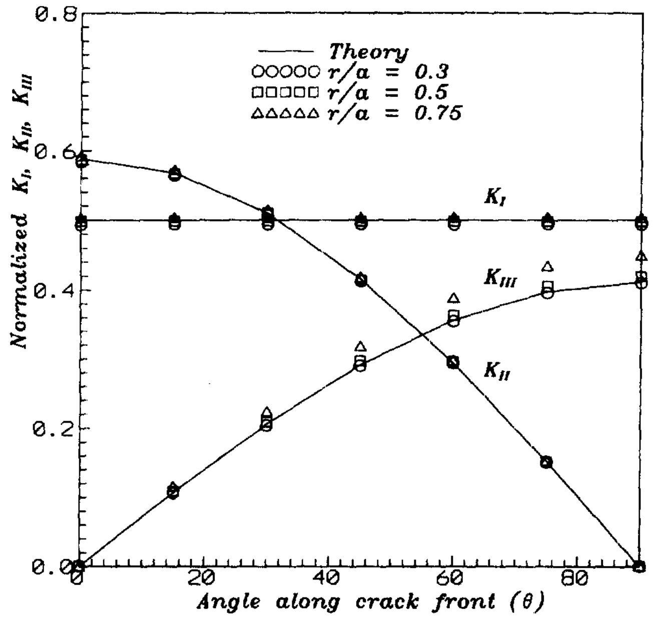

Fig. 6. Penny-shaped crack at 45° to tensile stress, fine mesh.

The normalized stress intensity factors K_{I}^*, K_{II}^*, K_{III}^* are plotted against angle along crack front θ for the crack inclined at angles 30° and 45° in Figs 5 and 6, respectively. This is for $r/a = 0.3$, 0.5 and 0.75 using a fine mesh. The K_{I}^*, K_{II}^*, K_{III}^* results for both the coarse and fine meshes and $r/a = 0.5$ are plotted against θ for $\alpha = 30$ and 45° in Figs 7 and 8, respectively.

By inspection of Table 3 and Figs 5 and 6 it is clear that the ratio $r/a = 0.75$ is too large for accurate K_{III} values. The contour is close to the centre of the penny-shaped crack and so the J-integral will be influenced by other parts of the crack front as well as the region of interest.

Good results were obtained for $r/a = 0.3$, 0.5 for both the coarse and fine meshes as can be seen from Table 3 and Figs 5–8. This shows that the implementation of the J-integral is robust for $r/a \leqslant 0.5$.

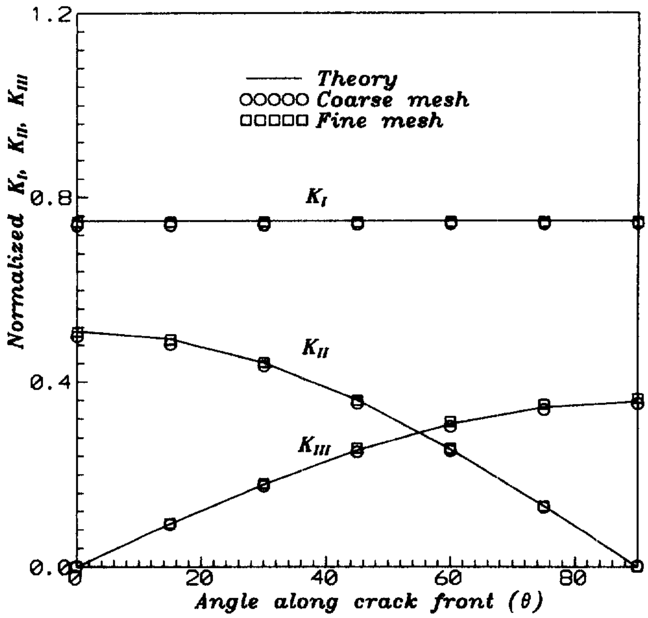

Fig. 7. Penny-shaped crack at 30° to tensile stress, $r/a = 0.5$.

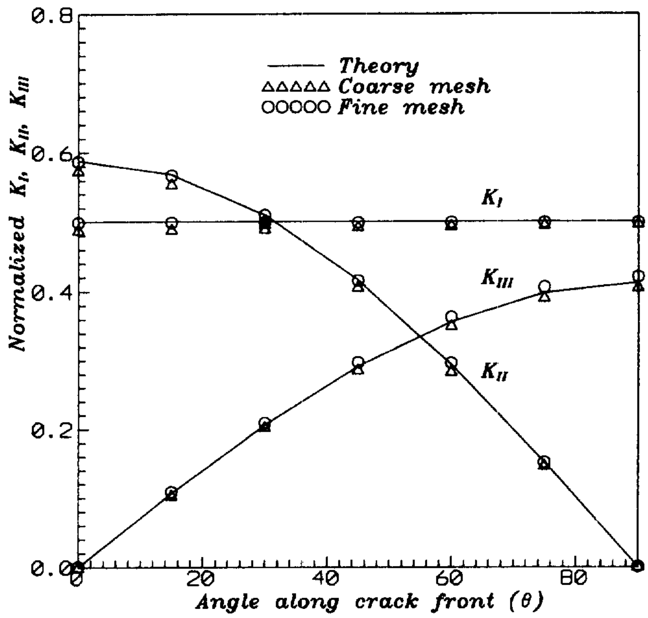

Fig. 8. Penny-shaped crack at 45° to tensile stress, $r/a = 0.5$.

The ratio of the area part of the J-integral to the contour part ($R_{\Omega/\Gamma}$) varies according to the radius of the J contour, and generally increases as r/a increases. For $r/a = 0.5$ the ratio $R_{\Omega/\Gamma}$ was 0.15, 0.11 and 0.285 for J^{I}, J^{II} and J^{III}, respectively, and there was no variation with θ or with α. However, $R_{\Omega/\Gamma}$ did vary with θ for J^{AS} with ranges $R^{\mathrm{AS}}_{\Omega/\Gamma} = 0.11$–$0.285$ for $\theta = 0°$ to $90°$, but did not vary with α. The total J-integral $R_{\Omega/\Gamma}$ varied with both θ and α and was $R^{\mathrm{tot}}_{\Omega/\Gamma} = 0.14$–$0.19$ for $\alpha = 30°$ and $R^{\mathrm{tot}}_{\Omega/\Gamma} = 0.13$–$0.225$ for $\alpha = 45°$ ($\theta = 0$–$90°$).

In conclusion the decomposition method gives good results for mode I, II and III stress intensity factors. The graphs in Figs 5 and 6 show that the J-integrals follow closely the theoretical stress intensity factors for the entire range $\theta = 0$–$90°$ for both $\alpha = 30$ and $45°$. Better results are obtained with a finer mesh as would be expected (see Fig. 8), however, the gain in accuracy in the stress intensity factors is not large.

5. SUMMARY

In this paper mode I, II and III stress intensity factors have been obtained using the J-integral technique applied to three-dimensional boundary elements. Mode II and III stress intensity factors have proved problematic in the past and so a rigorous derivation of the decomposition of the J-integral into its constituent modes has been carried out. During this derivation it was found that the decomposition equation used for stress by some authors [see e.g. Nikishkov and Atluri (1987a); Shivakumar and Raju (1990); Rigby and Aliabadi (1993); Huber *et al.* (1993)] was erroneous and that the area integral quoted previously for mode II and III J-integrals was incorrect [see e.g. Rigby and Aliabadi (1993); Huber *et al.* (1993)]. The path-area independency of mode I, II and III J-integrals was demonstrated. That the decomposition method has been successfully implemented was demonstrated by the penny-shaped crack example where the mode I, II and III stress intensity factors were all obtained to a good order of accuracy.

It should be noted that the best results are obtained if the contour radius is kept as small as possible ($r/a \leqslant 0.5$). This correlates with the observation that the J-integral should be used only in the vicinity of the crack front in three dimensions. If the remote contour C, used for the J-integral, is too large then it is influenced by other parts of the crack front and not just the position of interest [see Shivakumar and Raju (1990)].

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APPENDIX

Hartranft and Sih (1969) used an asymptotic series expansion in three dimensions to obtain the elastic fields along planes centred on points located on the crack front. The singular stresses at a point P specified by dimension l (see Fig. 9) in a plane normal to the crack front are given as:

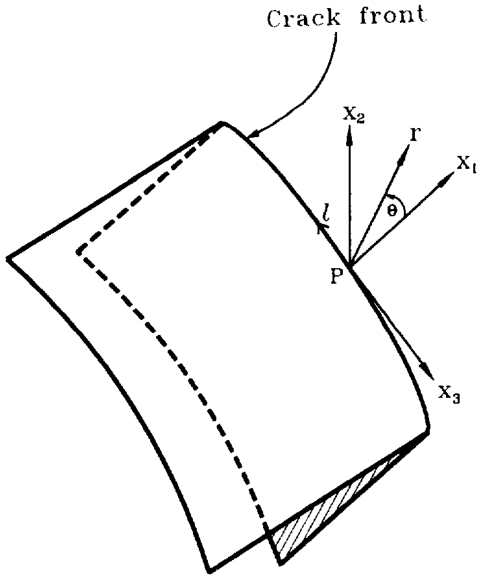

Fig. A1. Crack front axis system.

$$\sigma_{11} = \frac{K_{\rm I}(l)}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]-\frac{K_{\rm II}(l)}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left[2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] \tag{A1}$$

$$\sigma_{22} = \frac{K_{\rm I}(l)}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]+\frac{K_{\rm II}(l)}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \tag{A2}$$

$$\sigma_{33} = 2v\left[\frac{K_{\rm I}(l)}{\sqrt{2\pi r}}\cos\frac{\theta}{2}-\frac{K_{\rm II}(l)}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\right] \tag{A3}$$

$$\sigma_{12} = \frac{K_{\rm I}(l)}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}+\frac{K_{\rm II}(l)}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \tag{A4}$$

$$\sigma_{13} = -\frac{K_{\rm III}(l)}{\sqrt{2\pi r}}\sin\frac{\theta}{2} \tag{A5}$$

and

$$\sigma_{23} = \frac{K_{\rm III}(l)}{\sqrt{2\pi r}}\cos\frac{\theta}{2}. \tag{A6}$$

The next term in the expansion yields stresses which are constant with respect to r. In the crack coordinate system these stresses are:

$$\begin{aligned}(\sigma_{11})_2 &= 2C\\ (\sigma_{12})_2 &= 0\\ (\sigma_{13})_2 &= (A+B)\\ (\sigma_{22})_2 &= 0\\ (\sigma_{23})_2 &= 0\\ (\sigma_{33})_2 &= 2vC+(1+v)D\end{aligned} \tag{A7}$$

where $(\sigma)_2$ represents the second term in the series. A, B, C, D are functions of x_3 and are determined from the loading conditions of the problem under consideration.